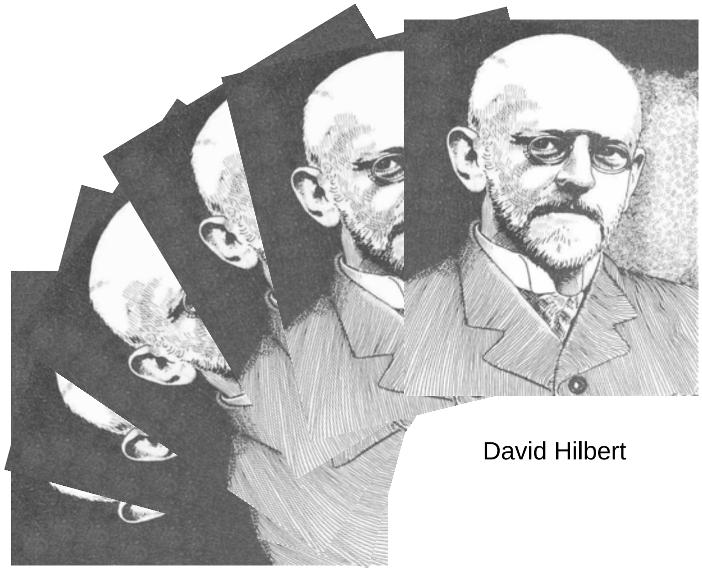
The Hilbert Transform



David Hilbert

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ABSTRACT: In this presentation, the basic theoretical background of the Hilbert Transform is introduced. Using this transform, normal real-valued time domain functions are made complex. This yields two useful properties - the Envelope and the Instantaneous Frequency. Examples of the practical use of these functions are demonstrated, with emphasis on acoustical applications.

Overview

- Definition
  - Time Domain
  - Frequency Domain
- Analytic Signals
  - Simple Example
- Applications
- MatLab Usage
  - Example: Muting Time
- Conclusion

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Overview of topics covered.

Hilbert Transform
What is it?

- Time Domain:
  $\lambda/4$ shift for all frequencies
- Frequency Domain:
  -90° phase shift for all spectral components

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The Hilbert Transform does not change domains. A Time Domain Function remains in the Time Domain and a Frequency Domain Function remains in the Frequency Domain. The effect is similar to an integration.

Hilbert Transform

$$H[a(t)] = \tilde{a}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} a(\tau)\frac{1}{t-\tau}\,d\tau$$

$$= \frac{1}{\pi}\,a(t) \;*\; \frac{1}{t}$$

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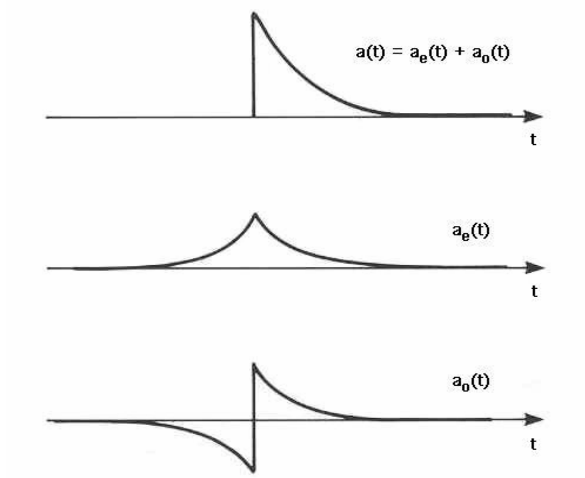
The Hilbert Transform in the Time Domain can be written as a convolution.

Even & Odd Functions

A causal time signal shown as a sum of an even signal and an odd signal



Even

Odd

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A few tools to understand the Hilbert Transform with a minimum of mathematics.

Fourier Transform Relationships

$a(t)$	$\overset{\mathcal{F}}{\Longleftrightarrow}$ $A(f) = R(f) + jX(f)$
Real and Even	Real and Even
Real and Odd	Imaginary and Odd
Real	*R* Even, *X* Odd
Imaginary	*R* Odd, *X* Even

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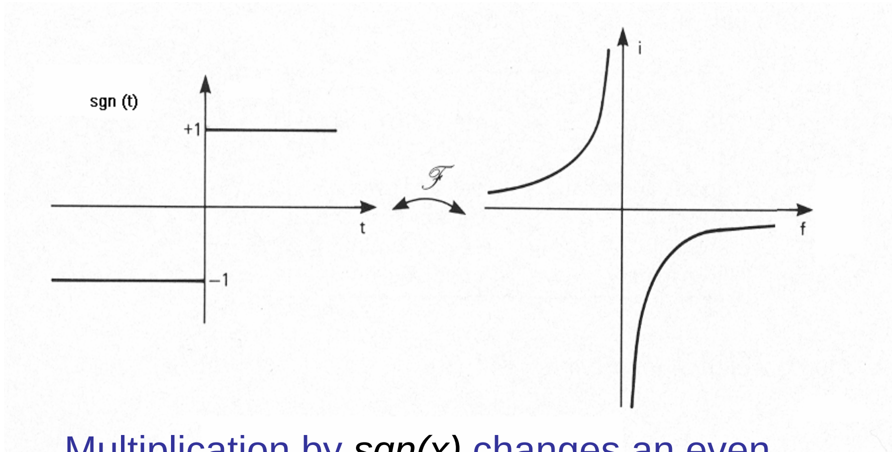
From the Symmetry Property of the Fourier Transform: F{F{a(t)}} = a(-t).

"Sign" Function



Multiplication by *sgn(x)* changes an even function to odd and vice-versa

The Fourier Transform of the sgn(x) function is 1/X.

Frequency Domain

$$F[\tilde{a}(t)] \equiv A(f)(-j \cdot \operatorname{sgn} f)$$

- 90° (-j) for positive frequencies
+90° (j) for negative frequencies

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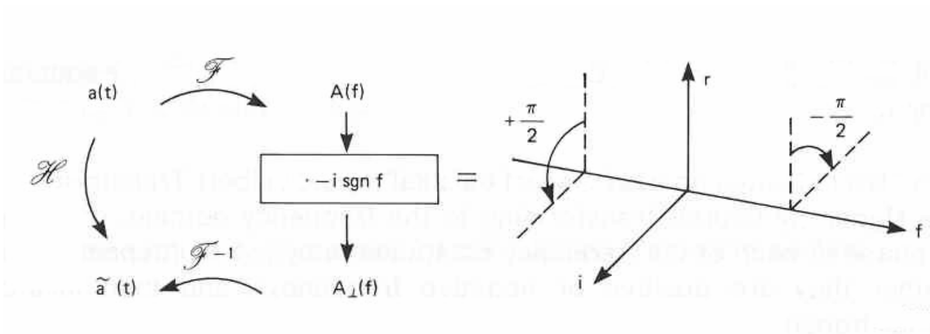
Using these tools, we can write the Hilbert Transform in the Frequency Domain as shown.

Effect of Hilbert Transform



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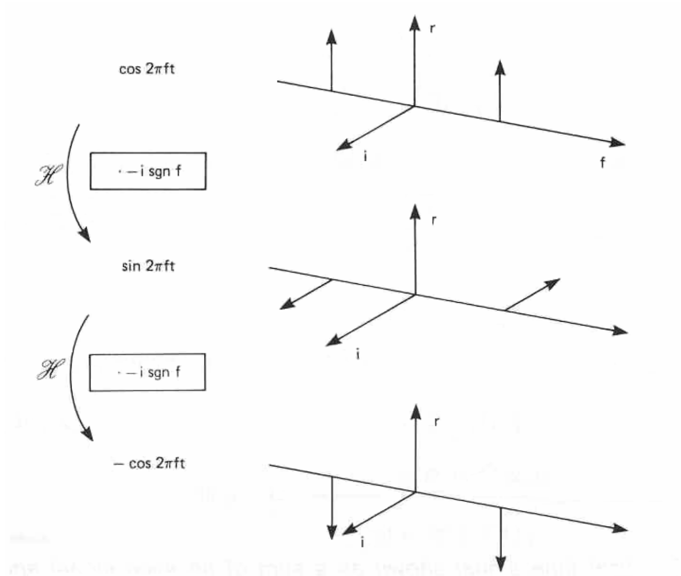
Effect of the Hilbert Transform in the Frequency Domain.

Hilbert Transform of a Sinusoid



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Successive Hilbert Transforms of a sinusoid.

Analytic Signal

$$\overset{\nabla}{a}(t) \equiv a(t) + j\,\tilde{a}(t)$$
$$= \left|\overset{\nabla}{a}(t)\right| \cdot e^{j\theta(t)}$$

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Definition of an Analytic Signal and the Envelope Function. The spectrum of the Analytic signal is is one-sided (positive only) and positive valued.

Analytic Signal - Descriptors

Magnitude

$$\left|\overset{\nabla}{a}(t)\right| = \sqrt{a^2(t) + \tilde{a}^2(t)}$$

Instantaneous Phase

$$\theta(t) = tan^{-1}\frac{\tilde{a}(t)}{a(t)}$$

Instantaneous Frequency

$$f_i(t) = \frac{1}{2\pi}\,\frac{d\theta(t)}{dt}$$

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Additional signal descriptors available from the Hilbert Transform and Analytic Signal.

Analytic Signal - Example

Real	$a(t) = \cos 2\pi f_0 t$
Imaginary	$\tilde{a}(t) = \sin 2\pi f_0 t$
Magnitude	$\left\lvert \overset{\nabla}{a}(t) \right\rvert = 1$
Phase	$\theta(t) = 2\pi f_0 t$
Instantaneous Frequency	$f_i(t) = f_0$

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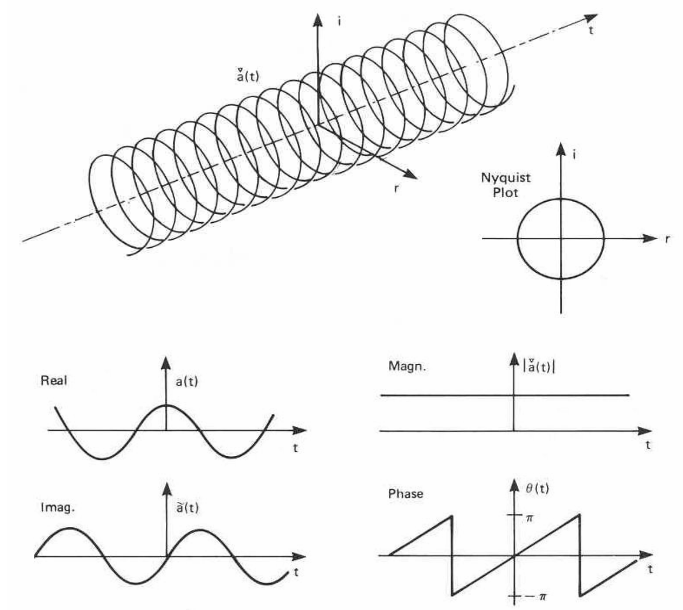
Example.

Analytic Signal



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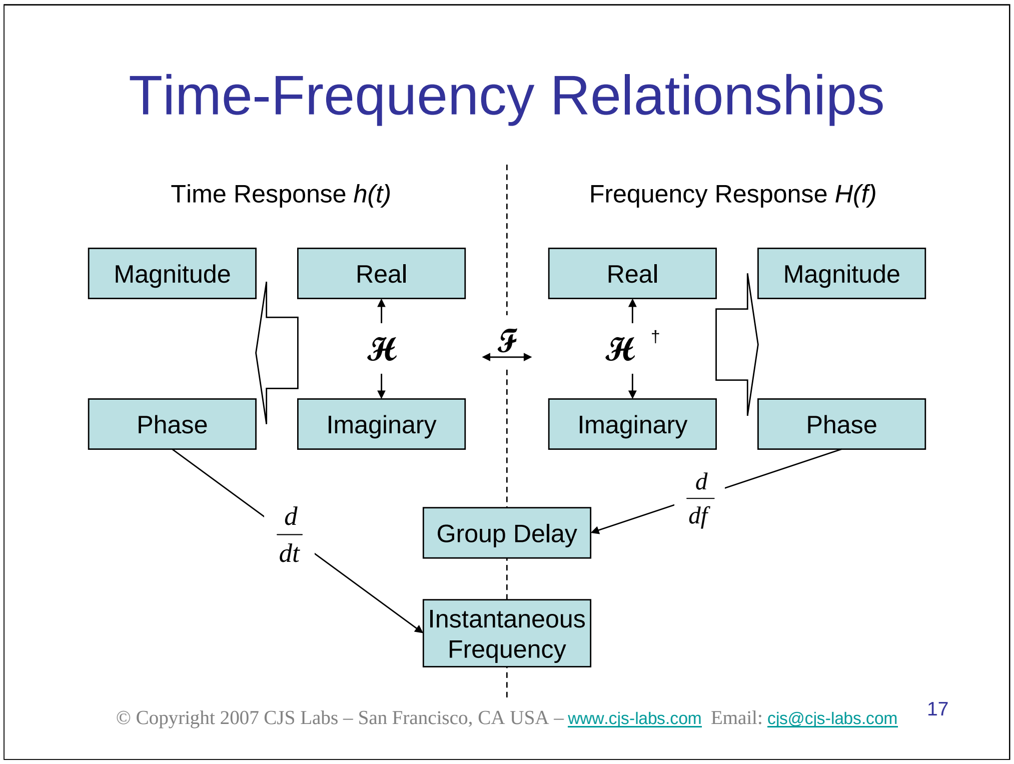
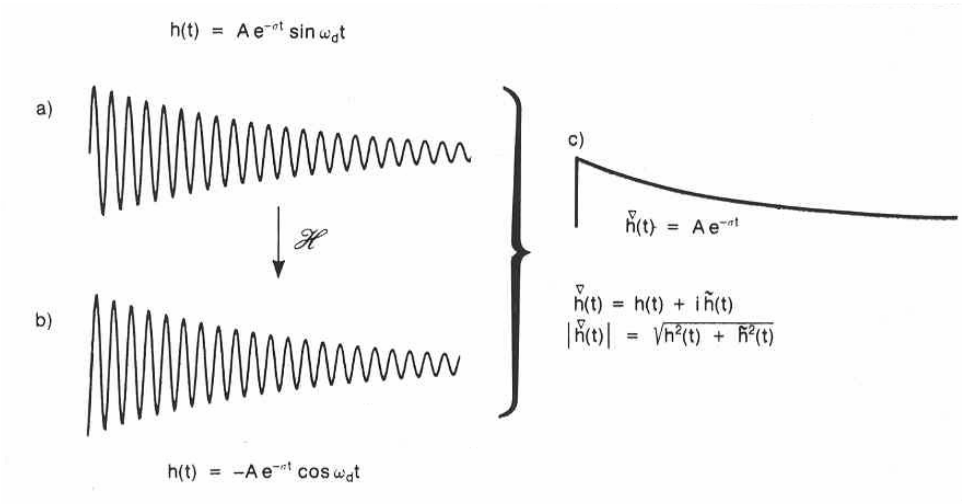
Sinusoidal Analytic Signal shown as a “Heyser Spiral”. The Nyquist Plot is the projection along the time axis. The Real and Imaginary parts are the projections along the real and imaginary axes, respectively.



Time & Frequency relationships of the various descriptors for a causal two-port system response function.

Envelope Extraction



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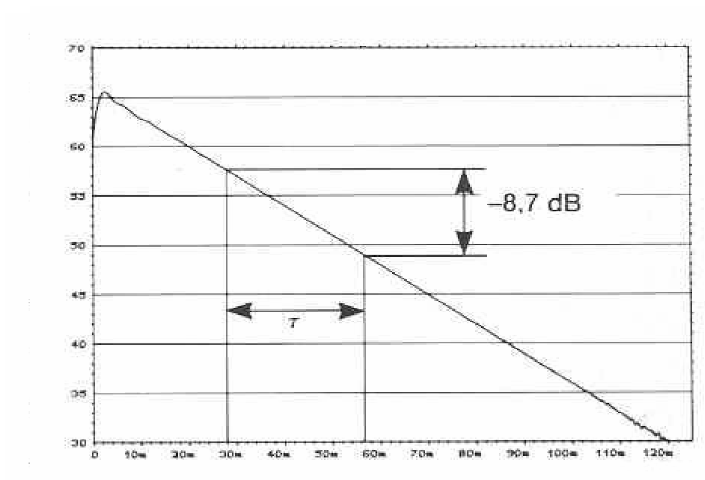
Application of the Hilbert Transform to Envelope extraction.

Decay Time Estimation (τ, T_{60})

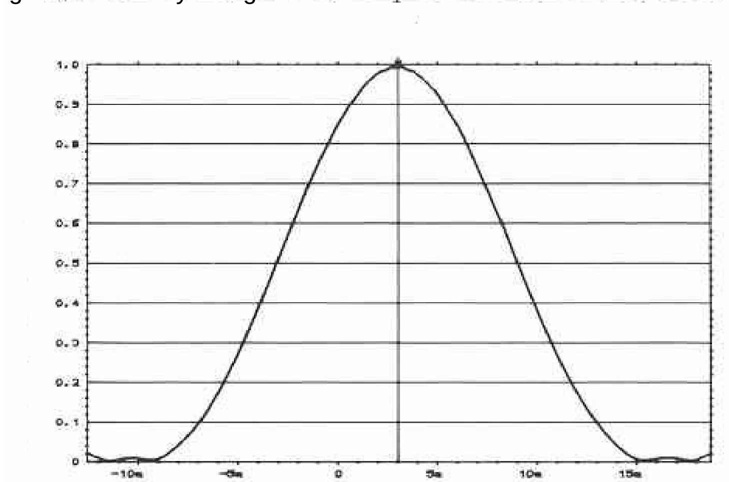
Exponentially Decaying Sinusoid
Magnitude of Analytic Signal



Application of the Hilbert Transform to Envelope extraction and decay time estimation, RT60 or RC LC circuit time constant, etc. The positive-valued magnitude function can be graphed on a log amplitude scale, enabling a far wider dynamic range than for a real-valued time signal.

Propagation Delay / Signal Arrival Estimation

Real Time Signal
Hilbert Transform of Signal
Magnitude of Analytic Signal - Peak indicates arrival time



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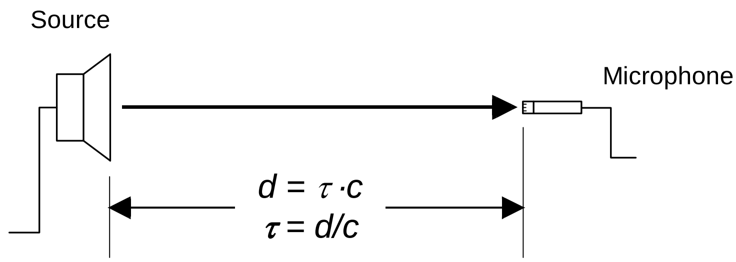
Application of the Hilbert Transform to Envelope to acoustic signal propagation time estimation.

Practical Example



$c = 344\ m/s$
$d = 1\ m$
$\tau = 2.92\ ms$

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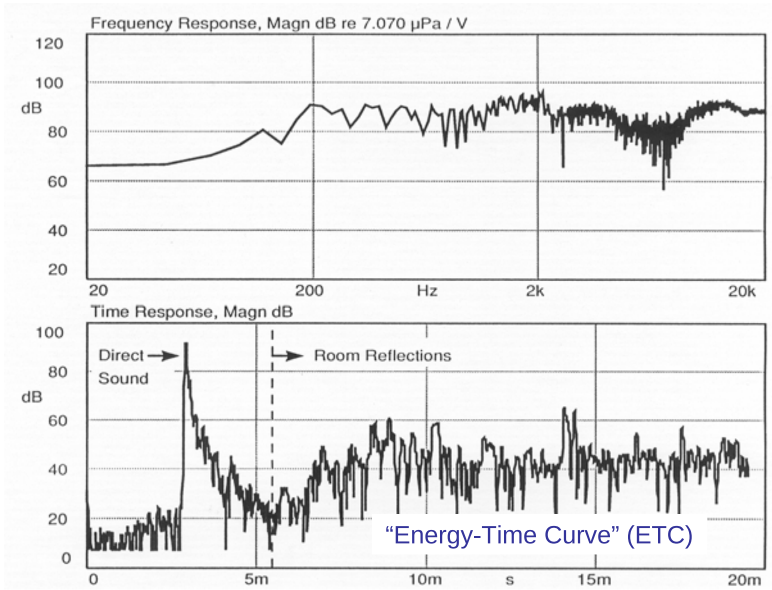
Example: Loudspeaker at 1 metre.

Loudspeaker Measurement



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Resulting response without windowing, shown as magnitude in both the Time and Frequency Domains.

All-Pass / Minimum Phase System Separation

$$H(f) = A(f)_{min} \cdot A(f)_{All\,Pass} \cdot e^{j(\phi(f)_{min} + \phi(f)_{All\,Pass})}$$

$$= A(f)_{min} \cdot e^{j(\phi(f)_{min} + \phi(f)_{All\,Pass})}$$

$$\phi(f)_{\min} = H^{-1}\left[\ \ln A(f)_{\min}\ \right]$$

For a minimum Phase system, it can be shown that the phase is not independent of the magnitude, but an be derived using the Hilbert Transform as shown. The All-Pass function has poles and zeroes that are negative conjugates of one-another, so the magnitude is unity. The phase of the All-Pass is a pure delay.

MatLab Usage

Y = HILBERT(X) computes the so-called discrete-time analytic signal

$$Y = \mathrm{Re}\{X\} + j \cdot \tilde{X}$$

where $\tilde{X}$ is the Hilbert transform of the vector $\mathrm{Re}\{X\}$.

Example: Mute Analysis

The HILBERT function in MATLAB.

Conclusion

Use of the Hilbert Transform:

- Allows definition of the *analytic signal* from a real-valued time signal:
  - The Real Part is the original time signal
  - The Imaginary Part is the *Hilbert Transform*
- This enables calculation of the *envelope* (magnitude) of a time signal
- Applications:
  - Envelope Extraction / Magnitude Estimation
  - Decay Time Estimation: RT_{60}, τ
  - Propagation Delay & Signal Arrival Measurements
  - All-Pass / Minimum Phase System Separation

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Conclusion.